Critical behavior of a one-dimensional diffusive epidemic process

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We investigate the critical behavior of a one-dimensional diffusive epidemic propagation process by means of a Monte Carlo procedure. In the model, healthy (A) and sick (B) individuals diffuse on a lattice with diffusion constants D_A and D_B, respectively. According to a Wilson renormalization calculation, the system presents a second-order phase transition between a steady reactive state and a vacuum state, with distinct universality classes for the cases $D_A = D_B$ and $D_A < D_B$. A first-order transition has been conjectured for $D_A > D_B$. In this work we perform a finite size scaling analysis of order parameter data at the vicinity of the critical point in dimension $d = 1$. Our results show no signature of a first-order transition in the case of $D_A > D_B$. A finite size scaling typical of second-order phase transitions fits well the data from all three regimes. We found that the correlation exponent $\nu = 2$ as predicted by field-theoretical arguments. Estimates for β/ν are given for all relevant regimes.

I. INTRODUCTION

The critical behavior of nonequilibrium reaction-diffusion systems describes relevant features of several phenomena in physics, chemistry, and biology, such as directed percolation, surface reactions, and epidemic propagation processes [1]. In general, these systems present a second-order phase transition between a vacuum state, where the order parameter density vanishes, to a steady reactive state [2–9,11]. At high dimensions where fluctuations on the particles densities can be neglected, these systems can be modeled by a set of mean-field-like differential equations. On the other hand, microscopic stochastic models defined on a lattice have provided a more accurate description of low-dimensional diffusion-limited reactions. Extensive numerical simulations [4,5] and analytical methods, such as mapping on exactly solvable quantum spin chains [6], real-space [7], field-theoretic [8,9], and Wilson renormalization group techniques [10] have shown that density fluctuations strongly modify the mean-field picture of a large class of such critical one-dimensional nonequilibrium kinetic models [11].

Recently, the propagation of an epidemic process in a population of fluctuating density has been studied using the Wilson renormalization group method [10]. The model can be considered as a two-species contactlike process [12–14]. In this model healthy and sick individuals (A and B particles) diffuse independently with diffusion constants D_A and D_B. Upon contact, sick individuals may infect healthy ones at a rate k_1. They also can spontaneously recover at a rate k_2. Therefore, a competition between the contamination process (creation of B particles) and the recovery process (annihilation of B particles) takes place. For low concentrations of the average total density ρ, the stationary state is characterized by a global extinction of the epidemics. Above a critical density ρ_c there is a stable steady-state regime with a fluctuating finite density of sick individuals. Near ρ_c the system exhibits a phase transition with the average density of sick individuals ρ_B acting as the order parameter.

This model can be formulated as a reaction-diffusion-decay process $A + B \rightarrow 2B$, $B \rightarrow A$. In a mean-field approach

it can be described by the set of partial nonlinear differential equations

$$\frac{\partial \rho_A}{\partial t} = D_A \nabla^2 \rho_A + k_2 \rho_B - k_1 \rho_A \rho_B , \qquad (1)$$

$$\frac{\partial \rho_B}{\partial t} = D_B \nabla^2 \rho_B - k_2 \rho_B + k_1 \rho_A \rho_B . \qquad (2)$$

For homogeneous initial conditions, the stationary state is easily found to consist of only A (healthy) particles below a total density threshold of $\rho_c^{mf} = (k_1/k_2)^{-1}$. Above this point, the density of B (sick) particles assumes a stationary value of $\rho_B = (\rho - \rho_c^{mf})^\beta$, with $\beta = 1$. The relaxation to the stationary state at criticality behaves asymptotically as $\rho_B \propto t^{-\beta/z\nu}$, with $\beta/z\nu = 1$.

However, fluctuations in the particle densities become relevant below the upper critical dimension $d_c = 4$ and corrections to the mean-field picture have to be introduced. The critical properties of the stationary state for the special case of $D_A = D_B$ falls in the same universality class studied by Kree *et al.* in the context of a population density in a polluted environment [15]. This process can also be characterized by the coupling between the fluctuations in the total density and the density of the species that is trying to survive. If fluctuations in the total density are suppressed, the transition falls in the universality class of directed percolation whose critical exponents are known with high accuracy in $d = 1$ to be $\nu = 1.097$, $\beta = 0.2769$, $z = 0.636$ ($\beta/z\nu = 0.3968$). Taking into account the total density fluctuations and using field-theoretic renormalization group techniques, Kree *et al.* [15] computed the critical exponents to be $\eta = -\epsilon/8$ in first order in $\epsilon = 4 - d$, $\nu = 2/d$ and $z = 2$ in all orders in ϵ.

In a recent work, van Wijland *et al.* [10] have shown that for $D_A < D_B$ the critical behavior is governed by a new fixed point. Within a Wilson renormalization group approach they found that the transition falls in a new universality class with exponents given by $\nu = 2/d$, $\eta = 0$ [$\beta = \nu(d + \eta)/2 = 1$] and $z = 2$ (in all orders in ϵ). In the opposite case of $D_A > D_B$ the

renormalization group equations do not have a fixed point and they have conjectured the possible existence of a fluctuation-induced first-order transition in this regime. Recent Monte Carlo simulations in $d=2$ have shown some signatures of the occurrence of such discontinuous transition in this regime [16].

The $d=1$ version of the symmetric $D_A=D_B$ case has been recently investigated in a Monte Carlo simulation by de Freitas *et al.* [17]. They reported $\beta/\nu=0.197(2)$. By fitting extrapolated order parameter data away from the critical point, where finite size and critical slowing down problems are less severe, they found $\beta=0.435(10)$ and $\nu=2.21(5)$. The reported value of β is quite above the directed percolation value. More importantly, the value of $\nu\neq 2$, indicates that higher-order symmetry breaking terms in the functional action seem to be relevant below dimension $d=2$ or, alternatively, that corrections to scaling are relevant for the system sizes simulated [18,19].

In this work we perform a Monte Carlo simulation of the above diffusion-reaction-decay process in $d=1$ for all three relevant regimes in order to give precise estimates of the equilibrium critical exponents and to check the conjectured emergence of a first-order transition for $D_A>D_B$. We will use a finite size scaling analysis of order parameter data to precisely locate the critical point and to directly compute the critical exponents β/ν and ν. These will be compared with previous Monte Carlo estimates, the ϵ-expansion renormalization group prediction and the $d=1$ directed percolation universality class.

II. MICROSCOPIC SIMULATION ALGORITHM AND RESULTS

In our simulations we initially distribute at random an equal number of A and B particles among the N sites of a chain with periodic boundary conditions. We consider each site as being a locus where an arbitrary number of particles can be located, i.e., the particles do not have a hard-core repulsive potential. The contamination process takes place only when A and B particles are found on the same site. The whole diffusion-reaction-decay process is done in three stages. In the first one we consider only the diffusive motion. Particles A and B are chosen to move with probabilities p_A and p_B, respectively (which take the role of the diffusion constants D_A and D_B). Once a particle is chosen to move, it
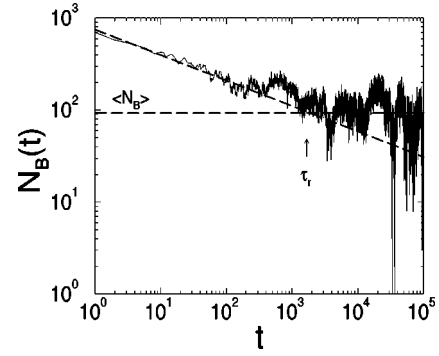

FIG. 1. A typical time evolution of the number of infected individuals near criticality. Data are from a chain of 400 sites, $p_A=0.5$, $p_B=0.25$, and $\rho=4.75$ (see Table I). The initial state has the same density of sick and healthy individuals. Notice that after being revived from an incidental fluctuation, the system returns to oscillate around a plateau. The relaxation time τ_r is estimated as the typical crossover time between the initial relaxation and the oscillatory behavior. Time is measured in units of lattice sweeps.

is displaced to one of the two neighboring sites with equal probability (unbiased diffusive motion). In the next stage we consider only the contamination process. Each A particle that is in the same site of at least one B particle will be transformed in a B particle with probability k_1 (the contamination rate). In the last stage each B particle can be replaced by an A particle with probability k_2 (recovering rate). Our simulations are performed using $k_1=k_2=1/2$. The time unit will be considered as the time needed to perform the above three stages over all particles.

At the steady state regime of finite size systems, ρ_B fluctuates around a plateau value for a very long time until an incidental fluctuation leads the system irreversibly to the vacuum state of $\rho_B=0$. These fluctuations become more frequent as the system approaches the critical density ρ_c. These incidental fluctuations towards the vacuum state are not present in the thermodynamical limit. Usually, averaging is limited to the surviving samples, as done in [17]. This procedure leads to an increase of statistical fluctuations with time due to the decrease of surviving samples, especially below the critical density where the plateau is very low in finite samples. In this work we choose to use a different approach to this problem. We make the absorbing process reversible by replacing a randomly chosen healthy individual

TABLE I. The present finite size scaling estimates of the critical concentration and the exponent ratio β/ν together with the renormalization group predictions. The absence of signatures of a first-order transition and the large value of β/ν found from the continuous transition hypothesis for $D_A>D_B$, together with the fact that $\beta\neq 1$ for $D_A<D_B$, indicate that higher-order terms on the action functional must be considered in the renormalization group analysis in $d=1$.

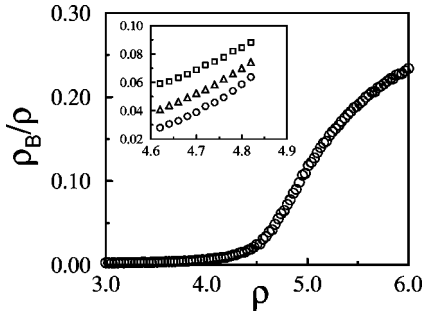| Diffusive regime	ρ_c	β/ν (Present MC)	β/ν (Field theory)
$D_A>D_B$ ($p_A=0.5$; $p_B=0.25$)	4.75(1)	0.336(15)	First-order transition
$D_A=D_B$ ($p_A=0.5$; $p_B=0.5$)	4.24(1)	0.226(20)	$\dfrac{d-\epsilon/8}{2}=0.3125$ in $d=1$
$D_A<D_B$ ($p_A=0.5$; $p_B=0.75$)	3.93(1)	0.165(22)	1/2 (all orders in ϵ expansion)

[a]References [10,15,16].

FIG. 2. The stationary value of ρ_B/ρ as a function of the total density for $p_A=0.5$ and $p_B=0.25$. For each concentration ρ, the stationary state was assumed to have been achieved after 12 800 time steps (lattice sweeps), which is around $8\tau_r$ near criticality. After that 400 consecutive microscopic configurations were used to account for some of the temporal fluctuations. Although the present method allows us to use the same run to proceed with the averaging process (leaving a convenient time interval between successive measurement blocks to generate uncorrelated configurations), we choose to average the whole procedure over 3200 distinct realizations. In the inset we show the size dependence of ρ_B/ρ at the vicinity of the critical point (from top to bottom $L=100,200,400$).

by a sick one whenever the system reaches the vacuum state. Therefore, within the same run the system returns its oscillatory behavior around the plateau and no runs have to be disregarded. A typical time evolution of the number of infected individuals near criticality is shown in Fig. 1. This procedure allows us to go well inside the vacuum phase which, in finite size simulations, appears as a long tail in the equilibrium density ρ_B. Usually, the average time between two consecutive incidental fluctuations towards the vacuum state is much larger than the relaxation time, except well inside the vacuum phase where both are of the order of a few time steps. The critical indices can be obtained after a finite size analysis is employed to achieve the proper thermodynamical limit. For the symmetric case of $p_A=p_B$, the critical index β/ν obtained by the present procedure is in agreement with, although slightly above (see Table I), the one reported in [17] where the surviving sampling technique was employed.

In Fig. 2 we show a typical plot of the average relative density of sick individuals ρ_B/ρ at the stationary regime versus the total density ρ for a system of linear size $N=400$, $p_A=0.5$, and $p_B=0.25$. For each concentration ρ, the stationary state was assumed to have been reached after 12 800 time steps (lattice sweeps), which is about eight times the relaxation time near criticality. For this case of $D_A>D_B$, the Wilson renormalization group results in first order in $\epsilon=4-d$ predict the emergence of a first-order transition. We searched for a hysteresis loop that could sign such discontinuous transition by performing very slow cycles of increase/decrease of the total density (two particles were increased/decreased at each cycle step). The average cycle presented no hysteresis loop within our numerical accuracy. However, the absence of hysteresis in Monte Carlo simulations of finite systems near a first-order transition can be due to jumps between the branches when long runs are considered. We further measured the equilibrium distribution func-
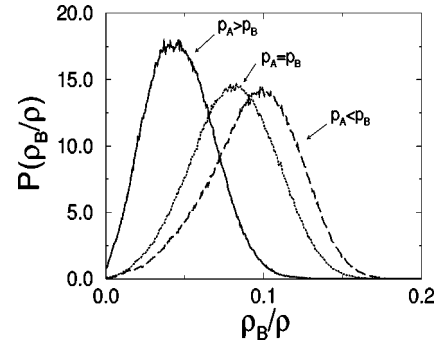

FIG. 3. The distribution function $P(\rho_B/\rho)$ versus ρ_B/ρ at criticality for $p_A=0.5$ and $p_B=0.25$ (continuous line), $p_B=0.5$ (dotted line), and $p_B=0.75$ (dashed line). The critical concentrations used are listed in the table. 10^6 consecutive configurations after relaxation were considered. The single peak structure points towards a continuous transition on all three regimes.

tion $P(\rho_B)$ near the transition where $P(\rho_B)d\rho_B$ is the fraction of equilibrium configurations with the density of sick individuals between ρ_B and $\rho_B+d\rho_B$. In the case of a well defined first-order transition $P(\rho_B)$ is expected to exhibit two peaks signaling the coexisting phases. However, in all three regimes $P(\rho_B)$ depicts a single peak structure (see Fig. 3). These results point out to the possibility of the active/inactive transition being also continuous for $D_A>D_B$ in $d=1$ or, alternatively, of it being a very weak first-order one. Assuming that the continuous transition scenario holds whatever the relation p_A/p_B, we will perform in the following a finite size scaling analysis of our data to estimate the critical indices of the emerging universality classes.

To precisely estimate the critical density ρ_c we assume that the order parameter ρ_B satisfies the scaling relation

$$\rho_B(\rho,L)=L^{-\beta/\nu}f[L^{1/\nu}(\rho-\rho_c)]. \qquad (3)$$

The preceding relation implies that the set of auxiliary functions

$$g(L,L',\rho)=\ln[\rho_B(L,\rho)/\rho_B(L',\rho)]/\ln(L/L') \qquad (4)$$

intersects in a single point $(\rho_c,\beta/\nu)$. In Fig. 4 we plot $g(L,L',\rho)$ for several lattice sizes and for the particular case of $p_A=0.5$ and $p_B=0.25$. From the intersection we estimate
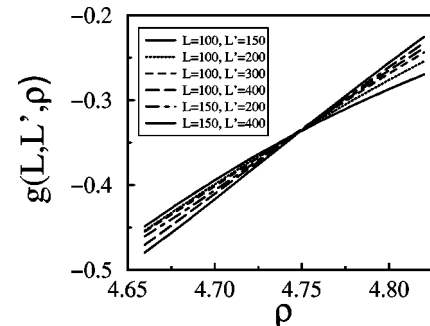

FIG. 4. The function $g(L,L',\rho)$ versus ρ for several pairs (L,L') and $p_A=0.5$, $p_B=0.25$. The common point determines ρ_c and β/ν.
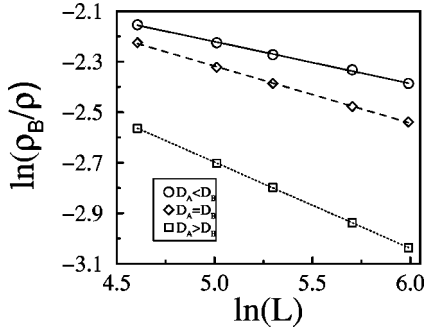
FIG. 5. $\ln(\rho_B/\rho)$ versus $\ln L$ at the critical point for the threes relevant regimes. The straight lines are the fits to $\rho_B/\rho|_{\rho_c} \propto L^{-\beta/\nu}$.
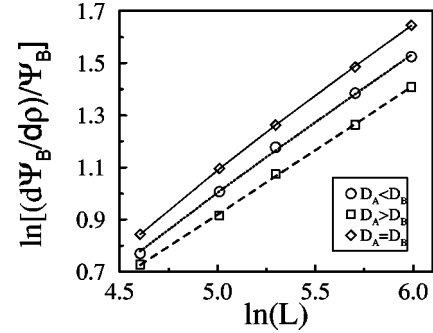

FIG. 6. $\ln[(1/\Psi_B)d\Psi_B/d\rho]$ versus $\ln L$ at the critical point for the three relevant regimes ($\Psi_B = \rho_B/\rho$). The lines are fits considering a first-order correction to scaling (see text). These corroborate the renormalization group prediction that $\nu = 2$.

the critical density $\rho_c = 4.75(1)$. Similar data were obtained for the cases of $p_B = 0.5$ and $p_B = 0.75$, giving rise, respectively, to $\rho_c = 4.24(1)$ and $\rho_c = 3.93(1)$. Error bars were estimated from the spread of the intersections of curves with all possible pairs (L, L'). Estimates for β/ν on all three cases can also be obtained from these intersections. The results are summarized in Table I where error bars include the error in the estimates of ρ_c. For the symmetric case the presently reported value of $\beta/\nu = 0.226(20)$ is slightly above the one reported by de Freitas *et al.*, $\beta/\nu = 0.197(2)$ [17], where an efficient algorithm specifically designed for this case was employed. Further intensive numerical work would be required to bring these two estimates more together. The actual order parameter data at ρ_c are shown in Fig. 5 as a function of the lattice size L. The data indicate that corrections to scaling in the order parameter are not relevant for the system sizes simulated. All three cases display clear power-law decays of $\rho_B(\rho_c, L)$ typical of continuous transitions.

The correlation exponent ν can be estimated by the size dependence of $(1/\Psi_B)d\Psi_B/d\rho|_{\rho_c} \propto L^{1/\nu}$, where $\Psi_B = \rho_B/\rho$. Our results are depicted in Fig. 6. The small curvature of the plotted data (especially for $D_A = D_B$) indicates that a small correction to scaling is present. In order to verify the renormalizaton group result that $\nu = 2/d$, we fitted our data considering a first-order correction to scaling $[(1/\Psi_B)d\Psi_B/d\rho|_{\rho_c} \propto L^{1/2}(1 + a_0/L)]$. The fits fully support that $\nu = 2$ for all three universality classes.

III. SUMMARY AND CONCLUSIONS

We have employed a finite size scaling analysis of Monte Carlo data for a diffusion-limited two-particle reaction that mimics the process of an epidemic propagation. A phase transition takes place between a steady state and a vacuum state as a function of the total particle density. Previous renormalization group results have shown that the critical

behavior depends of the relative mobility of the particles. A discontinuous transition is anticipated when sick individuals (B particles) diffuse slower than healthy ones (A particles) [10]. Previous Monte Carlo studies in $d = 2$ have shown some signatures of such first-order transitions [16]. A continuous transition occurs for $D_A \lesssim D_B$ with the correlation exponent $\nu = 2/d$ and the dynamical exponent $z = 2$. However, distinct universality classes in the symmetric ($D_A = D_B$) and asymmetric ($D_A < D_B$) cases are predicted. In particular the order parameter exponent β was found to reach the mean field value $\beta = 1$ for the asymmetric case in all orders in $\epsilon = 4 - d$.

The present results in $d = 1$ corroborate the renormalization group prediction that the correlation exponent $\nu = 2$. However, our results did not show any evidence of the conjectured first-order transition in the case of $D_A > D_B$. Although we cannot completely rule out the possibility of a very weak first-order transition, the data in this case also exhibit a finite size scaling typical of a second-order phase transition with a new class of critical exponents. Furthermore, for the case of $D_A < D_B$ the field-theory prediction of $\beta = 1$ is also not supported. These two results clearly indicate that the truncated higher-order terms on the action functional employed in the field-theoretical analysis are relevant in $d = 1$. It would be valuable to have these terms included in a future field-theoretical analysis of this low-dimensional system in order to have reliable analytical estimates for the full set of critical exponents on all relevant regimes.

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